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LETTER TO THE EDITOR

A set of new solutions of the perturbed S3 equation in a quasi-one-dimensional crystal

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Abstract. Mekhankov and Fedyanin have obtained several solutions for the perturbed S3 equation. We find the one-envelope-soliton solutions of the same equation by the Hirota bilinear method.

A wide class of quasi-one-dimensional systems of quantum statistical mechanics may be described to a good approximation by the following model Hamiltonian:

$$H = E_0 + p\sum_k N_k + \mu\sum_k (a_k^+ a_{k+1} + a_{k+1}^+ a_k) + \mu'\sum_k (a_k^+ a_{k+1}^+ - a_k a_{k+1}) + q\sum_k N_k N_{k+1} \qquad (1)$$

where $N_k = a_k^+ a_k$ and the basic equation for the operator $a_f(t)$ is

$$\begin{aligned} i\hbar \dot{a}_f(t) &= p a_f(t) + \mu[a_{f+1}(t) + a_{f-1}(t)] + \mu'[a_{f+1}^+(t) - a_{f-1}^+(t)] \\ &\quad + q[N_{f+1}(t) + N_{f-1}(t)]a_f(t). \end{aligned} \qquad (2)$$

One can define $\varphi_n(t)$ in the Heisenberg representation as follows (Mekhankov and Fedyanin 1984)

$$\varphi_n(t) = \langle 0|a_n(t)|0\rangle = \langle 0|V^+ a_n V|0\rangle$$

where

$$V = \lambda^{-1}\left(1 + \sum_n (\alpha_n(t)a_n^+ - \alpha_n^*(t)a_n)\right) \qquad a_n|0\rangle = 0.$$

When proceeding to the continuum limit $\varphi_n(t) \to \varphi(x, t)$, one easily obtains the perturbed S3 equation

$$i\hbar\dot{\varphi}(x, t) = (p + 2\mu)\varphi(x, t) + 2\mu' a_0 \varphi_x^*(x, t) + \mu a_0^2 \varphi_{xx}(x, t) + 2q|\varphi|^2\varphi(x, t) \qquad (3)$$

in which all the highest terms (with respect to non-linearity and dispersion) are dropped. Introducing dimensionless variables (τ, ξ)

$$\tau = \frac{\mu}{\hbar} t \qquad \xi = \frac{x}{a_0}$$

equation (3) becomes

$$i\varphi_\tau = \varphi_{\xi\xi} + \alpha\varphi + \beta\varphi_\xi^* + \gamma|\varphi|^2\varphi \qquad (4)$$

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where

$$\alpha = 2 + \frac{p}{\mu} \qquad \beta = \frac{2\mu'}{\mu} \qquad \gamma = \frac{2q}{\mu}.$$

We rewrite (4) in the form of a system of two equations for the real and imaginary parts of the wavefunction $\varphi = u + iv$

$$\begin{aligned} u_\tau &= v_{\xi\xi} + \alpha v - \beta v_\xi + \gamma(u^2+v^2)v \\ v_\tau &= -u_{\xi\xi} - \alpha u - \beta u_\xi - \gamma(u^2+v^2)u. \end{aligned} \tag{5}$$

We consider the dependent variable transformations

$$u = \frac{G(\xi, \tau)}{F(\xi, \tau)} \qquad v = \frac{H(\xi, \tau)}{F(\xi, \tau)} \qquad F = F^*, G = G^*, H = H^*. \tag{6}$$

Substituting these equations into (5), we have

$$\begin{aligned} &(D_\xi^2 - \alpha) F \cdot F = \gamma(G^2 + H^2) \\ &D_\tau G \cdot F = (D_\xi^2 - \beta D_\xi) H \cdot F \\ &D_\tau H \cdot F = (-D_\xi^2 - \beta D_\xi) G \cdot F. \end{aligned} \tag{7}$$

Here bilinear operators are defined by (Hirota 1973)

$$D_\xi^m D_\tau^n a(\xi, \tau) \cdot b(\xi, \tau) \equiv \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi'}\right)^m \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'}\right)^n a(\xi, \tau) b(\xi', \tau')\bigg|_{\xi=\xi',\, \tau=\tau'}.$$

Then, the one-envelope-soliton solution of equations (7) with $\alpha > 0$ and $\gamma < 0$ is given by

$$F = A_0 + A e^{\eta+\eta^*} \qquad G = B_0 + B e^{\eta+\eta^*} \qquad H = C_0 + C e^{\eta+\eta^*}$$

where $\eta = \kappa\xi - \Omega\tau + \eta^0$, k, Ω and η^0 are constants, and the real constants A_0, A, B_0, B, C_0 and C satisfy the following equations:

$$-\alpha A_0^2 = \gamma(B_0^2 + C_0^2)$$

$$[(\kappa+\kappa^*)^2 - \alpha] A_0 A = \gamma(B_0 B + C_0 C)$$

$$-\alpha A^2 = \gamma(B^2 + C^2)$$

$$(\Omega + \Omega^*)(B_0 A - A_0 B) = (\kappa+\kappa^*)^2 (A_0 C + C_0 A) - \beta(\kappa+\kappa^*)(A_0 C - C_0 A)$$

$$(\Omega + \Omega^*)(C_0 A - A_0 C) = -(\kappa+\kappa^*)^2 (A_0 B + B_0 A) - \beta(\kappa+\kappa^*)(A_0 B - B_0 A).$$

It is easy to obtain the simple solutions of (4) as follows:

$$\varphi(\xi, \tau) = \left(-\frac{\alpha}{\gamma}\right)^{1/2} \frac{\delta_1 + i\delta_2 G_0 \exp[\delta_3\sqrt{\alpha}\xi - (\delta_2/\delta_1)(\alpha - \delta_3\beta\sqrt{\alpha})\tau + \eta_0 + \eta_0^*]}{1 + G_0 \exp[\delta_3\sqrt{\alpha}\xi - (\delta_2/\delta_1)(\alpha - \delta_3\beta\sqrt{\alpha})\tau + \eta_0 + \eta_0^*]}$$

$$\varphi(\xi, \tau) = \left(-\frac{\alpha}{\gamma}\right)^{1/2} \frac{\delta_1 G_0 \exp[\delta_3\sqrt{\alpha}\xi + (\delta_2/\delta_1)(\alpha + \delta_3\beta\sqrt{\alpha})\tau + \eta_0 + \eta_0^*] + i\delta_2}{1 + G_0 \exp[\delta_3\sqrt{\alpha}\xi + (\delta_2/\delta_1)(\alpha + \delta_3\beta\sqrt{\alpha})\tau + \eta_0 + \eta_0^*]}$$

$$\varphi(\xi, \tau) = \left(-\frac{\alpha}{2\gamma}\right)^{1/2} (\delta_1 + i\delta_2) \frac{1 - G_0 \exp[\delta_3\sqrt{2\alpha}\xi - (\delta_2\delta_3/\delta_1)\beta\sqrt{2\alpha}\tau + \eta_0 + \eta_0^*]}{1 + G_0 \exp[\delta_3\sqrt{2\alpha}\xi - (\delta_2\delta_3/\delta_1)\beta\sqrt{2\alpha}\tau + \eta_0 + \eta_0^*]}$$

where G_0 is an arbitrary real constant, $\delta_j = +1$ or -1 ($j = 1, 2, 3$).

In conclusion, we have obtained the one-envelope-soliton solutions of the perturbed S3 equation in a quasi-one-dimensional crystal. The N-envelope-soliton solutions will be discussed in another paper.

References

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